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A Cone Decomposition Method for Optimal Contribution Selection in Forest Tree Management

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1 Optimal Contribution Selection

In a forest tree management, one of essential phases for tree improvement is on recurrent cycles of selection. In that phase, genetic diversity is a main consideration for genetic gain performance in the future. Therefore, an objective of optimal contribution selection (OCS) [1, 8, 11, 16] is to maximize the genetic benefit under a genetic diversity constraint by determining the gene contribution from each candidate.

This paper is concerned on OCS with an equal deployment problem (EDP) that designates a specified number of selected individuals to have equally contribution to the gene pool; and it can be formulated as:

$$\begin{aligned} \text{maximize} & : \mathbf{g}^T \mathbf{x} \\ \text{subject to} & : \mathbf{e}^T \mathbf{x} = 1, x_i \in \{0, \frac{1}{N}\} \ (i = 1, \dots, m), \mathbf{x}^T \mathbf{A} \mathbf{x} \leq 2\theta. \end{aligned} \quad (1)$$

Here, the objective is to maximize the total benefit $\mathbf{g}^T \mathbf{x}$ where $\mathbf{g} = (g_1, g_2, \dots, g_m)^T$ is the estimated breeding values (EBVs) [7] representing the genetic value of candidates in the gene contribution $\mathbf{x} \in \mathbb{R}^m$, and m is the total number of candidates. In this objective function, our decision variable is \mathbf{x} and we assume that \mathbf{g} is given. The first constraint $\mathbf{e}^T \mathbf{x} = 1$, with a vector of ones $\mathbf{e} \in \mathbb{R}^m$, demands unity of the total contribution from all candidates. The second constraint $x_i \in \{0, \frac{1}{N}\}$ interprets an equal contribution from each candidate, with N being the parameter to indicate the number of chosen candidates. Shortly speaking, N individuals has to be exactly chosen from a list of m available candidates in the EDP.

The last constraint in (1), $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 2\theta$, is our substantial constraint that requires a group coancestry $\frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{2}$ be under an appropriate level $\theta > 0$. If the group coancestry $\frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{2}$ is too high, the close relatedness among individuals in the population will decrease genetic diversity and also impact on the reduction of long-term genetic performance. The group coancestry [3] is computed with the Wright numerator relationship matrix [15] $\mathbf{A} \in \mathbb{R}^{m \times m}$; and [12] emphasized that \mathbf{A} is always symmetric and semi-definite positive.

In recent years, the OCS with EDP has been solved through a software package GENCONT [8] which is an implementation based on Lagrange multipliers, but it forcibly fixes variables that exceed lower or upper bound ($0 \leq x_i \leq \frac{1}{N}$) at the corresponding lower and upper bound. Thus, even

though GENCONT generates a solution quickly, the solution is often suboptimal. To resolve this difficulty in GENCONT, `dsOpt`, integrated in the software package OPSEL [9], was proposed by Mullin and Belotti[11]. Since `dsOpt` implements the branch-and-bound method combined with an outer approximation method [4], `dsOpt` generates a very large number of subproblems in the framework of branch-and-bound. This implementation is designed to acquire exact optimal solutions, but computing the solution takes a long time.

To deliver the problem in [8, 11], we consider to employ a second-order cone form into the quadratic constraint in 1. Utilizing Cholesky factorization of \mathbf{A} so that $\mathbf{A} = \mathbf{U}^T \mathbf{U}$, $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 2\theta$ can be reformulated:

$$\begin{aligned} \text{maximize} \quad & \mathbf{g}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{e}^T \mathbf{x} = 1, \quad x_i \in \left\{0, \frac{1}{N}\right\} \quad (i = 1, \dots, m), \quad \left(\sqrt{2\theta}N, \mathbf{U}\mathbf{x}\right) \in \mathcal{K}^m. \end{aligned} \quad (2)$$

with $\mathcal{K}^m = \{(v_0, \mathbf{v}) \in \mathbb{R}_+ \times \mathbb{R}^m : \|\mathbf{v}\|_2 \leq v_0\}$ is an $(m+1)$ -dimensional second-order cone. However, non-linearity arising from this second-order cone also leads to a heavy computation cost.

In this paper, we examine and propose a cone decomposition method (CDM) that is based on a geometric cut in a combination with a Lagrangian multiplier method and also draws on another form of second-order cones. A cutting plane is a geometric cut if the plane is computed with an orthogonal projection [2]. Cone decomposition itself has already been used in CPLEX which depends on an outer approximation, therefore, the proposed CDM generates a different linear approximation.

In addition, we prove that the Lagrangian multiplier method in the framework of CDM gives the analytical form for the geometric cut, therefore, the proposed CDM generates the linear cuts without relying on iterative methods.

The remainder of this paper is organized as follows. In Section 2, we propose the framework of CDM and demonstrates that the geometric cut in CDM has an analytical form. The numerical results will be presented in Section 3. Finally, in Section 4, we give some conclusions and discuss future studies.

2 Cone Decomposition Method

In this section, we focus on the proposed cone decomposition method (CDM) for EDP (2) that impose another form of second-order cones as in the following corollary [13].

Corollary 2.1. *A second-order cone \mathcal{K}^m can be also written as*

$$\mathcal{K}^m = \left\{ (v_0, \mathbf{v}) \in \mathbb{R}^{m+1} : \exists \mathbf{w} \in \mathbb{R}^m \text{ such that } v_j^2 \leq w_j v_0 \quad (j = 1, \dots, m), \quad \sum_{j=1}^m w_j \leq v_0, \quad v_0 \geq 0 \right\}.$$

Proof. Let $\hat{\mathcal{K}}^m$ be $\{(v_0, \mathbf{v}) \in \mathbb{R}^{m+1} : \exists \mathbf{w} \in \mathbb{R}^m \text{ s.t. } v_j^2 \leq w_j v_0 \quad (j = 1, \dots, m), \quad \sum_{j=1}^m w_j \leq v_0, \quad v_0 \geq 0\}$.

For $(\hat{v}_0, \hat{\mathbf{v}}) \in \hat{\mathcal{K}}^m$, if $\hat{v}_0 = 0$, then $\hat{\mathbf{v}} = \mathbf{0}$ due to the constraint $\hat{v}_j^2 \leq w_j \hat{v}_0$, therefore, we know $(\hat{v}_0, \hat{\mathbf{v}}) \in \mathcal{K}^m$. In the case $\hat{v}_0 > 0$, it holds that $\hat{v}_0 \geq \sum_{j=1}^m w_j \geq \sum_{j=1}^m \hat{v}_j^2 / \hat{v}_0$, and this leads to $\hat{v}_0 \geq \sqrt{\sum_{j=1}^m \hat{v}_j^2}$.

Conversely, we take $(v_0, \mathbf{v}) \in \mathcal{K}^m$. If $v_0 = 0$, we again have $\mathbf{v} = \mathbf{0}$; thus $(v_0, \mathbf{v}) \in \hat{\mathcal{K}}^m$. For positive v_0 , we can use $w_j = v_j^2 / v_0$ to show $(v_0, \mathbf{v}) \in \hat{\mathcal{K}}^m$. \square

Utilizing Corollary 2.1 and introducing new variable $\mathbf{y} = N\mathbf{x}$ to EDP (2) derives mixed-integer quadratic constraint problem (MI-QCP):

$$\begin{aligned} \text{maximize} \quad & : \frac{\mathbf{g}^T \mathbf{y}}{N} \\ \text{subject to} \quad & : \mathbf{e}^T \mathbf{y} = N, \mathbf{z} = \mathbf{U} \mathbf{y}, \\ & : z_i^2 \leq w_i c_0 \ (i = 1, \dots, m), \sum_{i=1}^m w_i \leq c_0, y_i \in \{0, 1\} \ (i = 1, \dots, m) \end{aligned} \quad (3)$$

where z_i is the i th element of \mathbf{z} , $c_0 = \sqrt{2\theta N^2}$, and the decision variables of our new formulation are \mathbf{y} , \mathbf{z} , and \mathbf{w} .

The nonlinear constraint in (3) is only the quadratic constraint $z_i^2 \leq w_i c_0$ with two variables z_i and w_i . In the proposed CDM, we generate the geometric cuts as cutting planes to these quadratic constraint by employing orthogonal projections [2]. Therefore, the framework of the proposed CDM is given as Algorithm 2.2.

Algorithm 2.2. [Cone decomposition method (CDM)]

Step 1 Let P^0 be an MI-LP problem that is generated from an optimization problem (3) by omitting the quadratic constraints $z_i^2 \leq w_i c_0$ ($i = 1, \dots, m$). Apply an MI-LP solver to P^0 , and let its optimal solution be $(\hat{\mathbf{y}}^0, \hat{\mathbf{z}}^0, \hat{\mathbf{w}}^0)$. Let $k = 0$.

Step 2 Let a set of generated cuts $\mathcal{C}^k = \emptyset$.

Step 3 For each $i = 1, \dots, m$, if $(\hat{z}_i^k)^2 \leq \hat{w}_i^k c_0$ is violated, apply the following steps.

Step 3-1 Compute the orthogonal projection of $(\hat{z}_i^k, \hat{w}_i^k)$ onto $z_i^2 \leq w_i c_0$ by solving the following sub-problem with the Lagrangian multiplier method;

$$\begin{aligned} \text{minimize} \quad & : \frac{1}{2} (\bar{z} - \hat{z}_i^k)^2 + \frac{1}{2} (\bar{w} - \hat{w}_i^k)^2 \\ \text{subject to} \quad & : \bar{z}^2 \leq \bar{w} c_0. \end{aligned} \quad (4)$$

Let $(\bar{z}_i^k, \bar{w}_i^k)$ be the solution of this subproblem.

Step 3-2 Add to \mathcal{C}^k the following linear constraint

$$\begin{pmatrix} \hat{z}_i^k - \bar{z}_i^k \\ \hat{w}_i^k - \bar{w}_i^k \end{pmatrix}^T \begin{pmatrix} z_i - \bar{z}_i^k \\ w_i - \bar{w}_i^k \end{pmatrix} \leq 0.$$

Step 4 If \mathcal{C}^k is empty, output $\hat{\mathbf{y}}^k$ as the solution and terminate.

Step 5 Build a new MI-LP P^{k+1} by adding \mathcal{C}^k to P^k . Let the optimal solution of P^{k+1} be $(\hat{\mathbf{y}}^{k+1}, \hat{\mathbf{z}}^{k+1}, \hat{\mathbf{w}}^{k+1})$.

Return to Step 2 with $k \leftarrow k + 1$.

Step 3-1 of Algorithm 2.2 computes the orthogonal projection. It would be desirable to compute the orthogonal projection on the original quadratic constraint $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 2\theta$, but such orthogonal projection does not have an analytic form. A numerical method has been proposed in [6], but it is an iterative method. Another iterative method is also proposed by [5] to solve different case of second-order cones. In contrast, the orthogonal projection in Step 3-1 is onto a specific cone $\bar{z}^2 \leq \bar{w} c_0$. The decomposition in Corollary 2.1 enables us to derive its analytical form, as proven in the following theorem.

Theorem 2.3. Assume that $(\hat{z}, \hat{w}) \in \mathbb{R}^2$ violates $\hat{z}^2 \leq \hat{w}c_0$. Let $(\bar{z}, \bar{w}) \in \mathbb{R}^2$ be the orthogonal projection of (\hat{z}, \hat{w}) onto $z^2 \leq wc_0$. Then, (\bar{z}, \bar{w}) can be given by an analytical form.

Proof. As in Step 3-1 of Algorithm 2.2, the orthogonal projection $(\bar{z}, \bar{w}) \in \mathbb{R}^2$ is the optimal solution of the subproblem (4) that has a convex closed feasible region. Since (\hat{z}, \hat{w}) is outside of the region $z^2 \leq wc_0$, the optimal solution of (4) can be obtained with the following problem:

$$\begin{aligned} \text{minimize} \quad & : \frac{1}{2}(z - \hat{z})^2 + \frac{1}{2}(w - \hat{w})^2 \\ \text{subject to} \quad & : z^2 = wc_0. \end{aligned} \quad (5)$$

Next, we define a Lagrangian function of (5) with a Lagrangian multiplier $\lambda \in \mathbb{R}$ as:

$$\mathcal{L}(z, w, \lambda) = \frac{1}{2}(z - \hat{z})^2 + \frac{1}{2}(w - \hat{w})^2 - \lambda(wc_0 - z^2).$$

We consider $\nabla \mathcal{L} = 0$ in the Lagrangian multiplier method, therefore we obtain the conditions below

$$z - \hat{z} + 2\lambda c_0 = 0; \quad w - \hat{w} - \lambda c_0 = 0; \quad -c_0 w + z^2 = 0$$

that results in a following cubic function with respect to λ :

$$4c_0^2\lambda^3 + (4c_0^2 + 4c_0\hat{w})\lambda^2 + (c_0^2 + 4c_0\hat{w})\lambda + (c_0\hat{w} - (\hat{z})^2) = 0.$$

When we apply Cardano's Formula [14] to this cubic function, we obtain only one real root $\bar{\lambda}$. This leads to $\bar{z} = \hat{z} - 2\bar{\lambda}c_0$ and $\bar{w} = \hat{w} + \bar{\lambda}c_0$. Therefore, the optimal solution (\bar{z}, \bar{w}) of (4) has an analytical form. \square

The termination of the proposed method is guaranteed by the following theorem.

Theorem 2.4. Algorithm 2.2 terminates in a finite number of iterations.

Proof. Due to the binary constraints $y_i \in \{0, 1\}$ for $i = 1, \dots, m$, the number of solution candidates is at most 2^m . In the proposed method, we remove at least one candidate, therefore, the number of iterations is bounded above by 2^m . \square

3 Numerical Experiment

Numerical experiments were conducted for performance comparison of the proposed method CDM, with the existing software **dsOpt** (as integrated in **OPSEL**) and **GENCONT**, and a general MI-SOCP solver **CPLEX**. We implemented CDM in **MATLAB 9.3.0.713579 (R2017b)** by setting **CPLEX 12.71** as the MI-LP solver. All numerical experiments were performed on Intel(R) Xeon(R) CPU E3-1231 (3.40 GHz) and 8 GB memory space under 64-bit Windows 10 operating system. The generated data by the simulation **POPSIM** [10] were taken from <https://doi.org/10.5061/dryad.9pn5m>. The sizes of the test instances are $m = 200, 1050, 2045, 5050, 10100$, and 15222. We set parameter $N = 50, 100$, and as a stopping criterion for **CPLEX**, we used gap = 1%, 5%. The computation time was limited to 3 hours for each execution.

Table 1 shows the results from the OCS solver **GENCONT**. In this table, the first, second, and third columns are the given parameter N , number of candidates m , and 2θ . The columns " $\mathbf{g}^T \mathbf{x}$ " and " $\mathbf{x}^T \mathbf{A} \mathbf{x}$ " are the obtained objective values and group coancestry, respectively. Table 1 shows

Table 1: The result from GENCONT

N	m	2θ	$\mathbf{g}^T \mathbf{x}$	$\mathbf{x}^T \mathbf{A} \mathbf{x}$	time (sec)	# chosen N
50	200	0.0334	11.472	0.03340	3.54	64
	1050	0.0627	25.91	0.06270	7.20	81
	2045	0.0711	438.36	0.07109	111.52	71
	5050	0.1081	43.44	0.10810	1561.43	78
100	200	0.0258	8.89	0.02580	0.48	93
	1050	0.0539	24.07	0.0539	4.77	94
	2045	0.0628	432.75	0.06279	106.48	74
	5050	0.0994	42.08	0.09940	1533.31	81

the computation time in the sixth column; and the number of chosen candidates (# chosen N) by GENCONT in the last column. For a feasible solution \mathbf{x} , it should hold $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 2\theta$ and the number of chosen candidate should be exactly N . However, the numerical result shows that # chosen N did not match the given N so that GENCONT failed to obtain feasible solutions. In addition, insufficient memory involved failure to output solution for $m > 5050$.

Tables 2 and 3 shows feasible solutions of the other methods for the given parameter $N = 50$ and $N = 100$, respectively. Contrary to GENCONT, the number of chosen candidates of the rest methods match the given N . the first column displays different methods for our numerical experiment. In both tables, CPLEX-default means that we used the default setting in CPLEX, and CPLEX-LPrelex means that we explicitly set a parameter so that CPLEX used LP relaxation forcibly. We set the time limit of 3 hours, and we indicate the violations of this time limit by '> 3 hours' and we used the best objective values in the 3 hours. In the case of out of memory, we used "OOM."

CPLEX-default shows its computation efficiency when the gap (the stopping criterion) is 5%. On the other hand, for larger problems or smaller gaps, CPLEX-default is more time-consuming than other methods. For instance, we can see a large time difference for the smallest size $m = 200$. CPLEX-default for gap 5% is the most efficient method among the seven methods, but it turns to be the slowest method when we set the gap as 1%. For such a tight gap, CPLEX-LPrelex and CDM can reduce computation time to less than 5 seconds. In the case $m = 15222$, CPLEX-default could not finish its computation within the time limit (three hours), and the best objective value in the three hours was much worse than CPLEX-LPrelex and CDM; CPLEX-default only reached $\mathbf{g}^T \mathbf{x} = 107.56$.

From the difference between the results of CPLEX-default and those of CPLEX-LPrelex, we can infer that the default setting of CPLEX cannot solve EDPs efficiently, and we have to explicitly let CPLEX know that LP relaxation is effective for EDPs.

Table 3 shows the results for the case $N = 100$. Similar with the result in the previous table that CPLEX-LPrelex and CDM obtain feasible solutions without having a memory problem. However, when our proposed method CDM is compared with CPLEX-LPrelex, CPLEX-LPrelex gives better computation time performance than CDM for $m \leq 10100$. This is not only shown by Table 3, but also it is on Table 2. For example, the computation time of CDM is 5 times slower than CPLEX-LPrelex to generate the solution of OCS with $m = 10100$. Only for the largest size problem $m = 15222$, CDM can show its efficiency among all methods.

Table 2: Numerical comparison for EDPs ($N = 50$)

Method	m	2θ	gap = 5%			gap = 1%		
			$\mathbf{g}^T \mathbf{x}$	$\mathbf{x}^T \mathbf{A} \mathbf{x}$	time (sec)	$\mathbf{g}^T \mathbf{x}$	$\mathbf{x}^T \mathbf{A} \mathbf{x}$	time (sec)
CPLEX-default	200	0.0334	24.99	0.03340	1.06	25.19	0.03340	8735.24
CPLEX-LPrelax			25.16	0.03340	1.96	25.19	0.03340	3.96
dsOpt			25.12	0.03340	5.32	25.18	0.03340	606.94
CDM			25.02	0.03340	1.69	25.15	0.03340	1.72
CPLEX-default	1050	0.0627	24.97	0.06267	3.56	24.97	0.06267	6.64
CPLEX-LPrelax			24.94	0.06265	4.27	24.94	0.06265	4.64
dsOpt			24.97	0.06169	5.19	24.85	0.06268	> 3 hours
CDM			24.65	0.06118	9.41	24.96	0.06238	12.11
CPLEX-default	2045	0.0711	437.21	0.07100	3.95	437.21	0.07100	3.83
CPLEX-LPrelax			438.07	0.07060	2.97	438.08	0.07060	3.52
dsOpt			432.94	0.06700	7.09	435.87	0.07020	14.42
CDM			434.26	0.06760	1.76	437.38	0.06960	2.52
CPLEX-default	5050	0.1081	41.90	0.10776	73.16	42.57	0.10781	> 3 hours
CPLEX-LPrelax			42.46	0.10658	11.42	42.46	0.10658	15.19
dsOpt			41.57	0.10471	236.70	42.67	0.10807	> 3 hours
CDM			42.56	0.10742	187.24	42.56	0.10742	182.10
CPLEX-default	10100	0.0701	44.89	0.06931	> 3 hours	44.89	0.06931	> 3 hours
CPLEX-LPrelax			45.91	0.06789	104.44	46.48	0.07008	200.55
dsOpt			46.00	0.07005	4509.83	46.21	0.06975	8787.37
CDM			45.27	0.06896	1003.67	46.43	0.07005	1204.47
CPLEX-default	15222	0.0388	118.33	0.03840	> 3 hours	107.56	0.03280	> 3 hours
CPLEX-LPrelax			454.07	0.03860	350.14	458.85	0.03880	1080.17
dsOpt					OOM			OOM
CDM			452.57	0.03880	450.84	461.83	0.03880	547.02

4 Conclusion and Future Work

In this study, we proposed the implementation of cone decomposition method to optimal contribution selection in forest tree management. The computation time problem difficulty from OPSEL makes us consider to propose the efficiency methods for solving OCS. We compared the efficiency of our proposed implementation with the existing breeding selection software (GENCONT and OPSEL) and also with the optimization solver CPLEX.

Based on the numerical result, we observed that our proposed relaxations, CDM still needs further improvement. It is seen by comparing CDM with CPLEX-LPrelax that CDM can only give better performance than CPLEX-LPrelax on the largest size problem. Therefore, in future study, we want to implement a sparsity structure on CDM so that it can reduce the computation time problem.

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Table 3: Numerical comparison for EDPs ($N = 100$)

Method	m	2θ	gap = 5%			gap = 1%		
			$\mathbf{g}^T \mathbf{x}$	$\mathbf{x}^T \mathbf{A} \mathbf{x}$	time (sec)	$\mathbf{g}^T \mathbf{x}$	$\mathbf{x}^T \mathbf{A} \mathbf{x}$	time (sec)
CPLEX-default	200	0.0258	23.19	0.02580	4.31	23.49	0.02580	13.14
CPLEX-LPrelax			23.52	0.02580	3.08	23.52	0.02580	3.54
dsOpt			23.14	0.02575	1.30	23.54	0.02580	566.89
CDM			23.53	0.02580	1.78	23.55	0.02580	2.03
CPLEX-default	1050	0.0539	22.53	0.05389	6.68	22.53	0.05389	3.64
CPLEX-LPrelax			22.55	0.05371	8.10	22.55	0.05371	5.61
dsOpt			21.79	0.05358	6.07	22.25	0.05382	193.08
CDM			22.49	0.05339	17.02	22.49	0.05339	15.23
CPLEX-default	2045	0.0628	420.04	0.06100	3.21	420.04	0.06100	3.08
CPLEX-LPrelax			420.79	0.06190	4.28	420.79	0.06190	3.08
dsOpt			419.53	0.06155	7.93	419.53	0.06155	7.96
CDM			418.67	0.06010	2.56	418.67	0.06010	2.43
CPLEX-default	5050	0.0994	40.63	0.09932	58.37	40.63	0.09932	54.43
CPLEX-LPrelax			40.56	0.09868	27.22	40.56	0.09868	19.23
dsOpt			40.13	0.09860	134.55	40.47	0.09936	367.29
CDM			40.28	0.09821	183.56	40.35	0.09742	197.38
CPLEX-default	10100	0.0610	43.79	0.06059	2720.18	44.34	0.06070	> 3 hours
CPLEX-LPrelax			44.43	0.06061	197.51	44.42	0.06061	216.66
dsOpt			43.36	0.06018	584.77	44.44	0.06100	7538.99
CDM			43.86	0.06095	948.07	44.53	0.06092	1282.72
CPLEX-default	15222	0.0300	436.92	0.02990	5084.69	436.92	0.02990	> 3 hours
CPLEX-LPrelax			423.75	0.02985	603.78	438.96	0.03000	710.21
dsOpt					OOM			OOM
CDM			432.13	0.02865	632.34	439.88	0.02960	448.82

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